

Entanglement monotones for multi-qubit states based on geometric invariant theory

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We construct entanglement monotones for multi-qubit states based on Plücker coordinate equations of Grassmann variety, which are central notion in geometric invariant theory. As an illustrative example, we in details investigate entanglement monotones of a three-qubit state.

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I. INTRODUCTION

The geometry of entanglement is a very interesting subject as much as quantification and classification of entanglement are [1, 2, 3, 4, 5, 6]. It is possible to derive geometrical measures of entanglement invariant under Stochastic Local quantum Operation and Classical Communication (SLOCC). All homogeneous positive functions of pure states that are invariant under determinant-one SLOCC operations are entanglement monotones [7]. In this paper, we will derive entanglement monotones based on a branch of the algebraic geometry called geometric invariant theory. In particular, let G be a group that acts on a set A , then the invariant theory is concerned with the study of the fixed points A^G and the orbits A/G associated to this action. The geometric invariant theory deals with the case where G is an algebraic group, e.g., a special linear group $SL(r, \mathbf{C})$, that acts on a variety A via morphisms. Thus, based on the geometric invariant theory, we can construct a measure of entanglement that is invariant under action of $SL(r, \mathbf{C})$ by construction. It has a well defined geometrical structure called Grassmann variety or Grassmannian and it is generated by a quadratic polynomials called the Plücker coordinate equations. We will in detail discuss our construction in the following section. Recently, Péter Lévy [8] has constructed a class of multi-qubit entanglement monotones, which was based on the construction of C. Emary [9]. His construction based on bipartite partitions of the Hilbert space and the invariants was expressed in terms of the Plücker coordinates of the Grassmannian. However, we do have different approaches and construction to solve the problem of quantifying multipartite states, but some of the result on entanglement monotones for multi-qubit states coincide. Now, let us start by denoting a general, pure, composite quantum system with m subsystems $\mathcal{Q} = \mathcal{Q}_m^p(N_1, N_2, \dots, N_m) = \mathcal{Q}_1 \mathcal{Q}_2 \cdots \mathcal{Q}_m$, consisting of the pure state $|\Psi\rangle = \sum_{k_1=1}^{N_1} \sum_{k_2=1}^{N_2} \cdots \sum_{k_m=1}^{N_m} \alpha_{k_1, k_2, \dots, k_m} |k_1, k_2, \dots, k_m\rangle$ and corresponding the Hilbert space

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$\mathcal{H}_{\mathcal{Q}} = \mathcal{H}_{\mathcal{Q}_1} \otimes \mathcal{H}_{\mathcal{Q}_2} \otimes \cdots \otimes \mathcal{H}_{\mathcal{Q}_m}$, where the dimension of the j th Hilbert space is given by $N_j = \dim(\mathcal{H}_{\mathcal{Q}_j})$. We are going to use this notation throughout this paper. In particular, we denote a pure two-qubit state by $\mathcal{Q}_2^p(2, 2)$. Next, let $\rho_{\mathcal{Q}}$ denote a density operator acting on $\mathcal{H}_{\mathcal{Q}}$. The density operator $\rho_{\mathcal{Q}}$ is said to be fully separable, which we will denote by $\rho_{\mathcal{Q}}^{sep}$, with respect to the Hilbert space decomposition, if it can be written as $\rho_{\mathcal{Q}}^{sep} = \sum_{k=1}^N p_k \bigotimes_{j=1}^m \rho_{\mathcal{Q}_j}^k$, $\sum_{k=1}^N p_k = 1$ for some positive integer N , where p_k are positive real numbers and $\rho_{\mathcal{Q}_j}^k$ denotes a density operator on Hilbert space $\mathcal{H}_{\mathcal{Q}_j}$. If $\rho_{\mathcal{Q}}^p$ represents a pure state, then the quantum system is fully separable if $\rho_{\mathcal{Q}}^p$ can be written as $\rho_{\mathcal{Q}}^{sep} = \bigotimes_{j=1}^m \rho_{\mathcal{Q}_j}$, where $\rho_{\mathcal{Q}_j}$ is the density operator on $\mathcal{H}_{\mathcal{Q}_j}$. If a state is not separable, then it is said to be an entangled state.

The general references for the complex projective space are [10, 11]. So, let $\{f_1, f_2, \dots, f_q\}$ be continuous functions $\mathbf{K}^n \rightarrow \mathbf{K}$, where \mathbf{K} is the field of real \mathbf{R} or complex numbers \mathbf{C} . Then we define real (complex) space as the set of simultaneous zeroes of the functions

$$\mathcal{V}_{\mathbf{K}}(f_1, f_2, \dots, f_q) = \{(z_1, z_2, \dots, z_n) \in \mathbf{K}^n : f_i(z_1, z_2, \dots, z_n) = 0 \ \forall \ 1 \leq i \leq q\}. \quad (1)$$

These real (complex) spaces become topological spaces by giving them the induced topology from \mathbf{K}^n . Now, if all f_i are polynomial functions in coordinate functions, then the real (complex) space is called a real (complex) affine variety. A complex projective space \mathbf{CP}^n is defined to be the set of lines through the origin in \mathbf{C}^{n+1} , that is, $\mathbf{CP}^n = (\mathbf{C}^{n+1} - 0)/\sim$, where \sim is an equivalence relation define by $(x_1, \dots, x_{n+1}) \sim (y_1, \dots, y_{n+1}) \Leftrightarrow \exists \lambda \in \mathbf{C} - 0$, such that $\lambda x_i = y_i \ \forall \ 0 \leq i \leq n$. For $n = 1$ we have a one dimensional complex manifold \mathbf{CP}^1 , which is a very important one, since as a real manifold it is homeomorphic to the 2-sphere \mathbf{S}^2 e.g., the Bloch sphere. Moreover, every complex compact manifold can be embedded in some \mathbf{CP}^n . In particular, we can embed a product of two projective spaces into the third one. Let $\{f_1, f_2, \dots, f_q\}$ be a set of homogeneous polynomials in the coordinates $\{\alpha_1, \alpha_2, \dots, \alpha_{n+1}\}$ of \mathbf{C}^{n+1} . Then the projective variety is defined to be the subset

$$\mathcal{V}(f_1, f_2, \dots, f_q) = \{[\alpha_1, \dots, \alpha_{n+1}] \in \mathbf{CP}^n : f_i(\alpha_1, \dots, \alpha_{n+1}) = 0 \ \forall \ 1 \leq i \leq q\}. \quad (2)$$

We can view the complex affine variety $\mathcal{V}_{\mathbf{C}}(f_1, f_2, \dots, f_q) \subset \mathbf{C}^{n+1}$ as a complex cone over the projective variety $\mathcal{V}(f_1, f_2, \dots, f_q)$. We can also view \mathbf{CP}^n as a quotient of the unit $2n + 1$ sphere in \mathbf{C}^{n+1} under the action of $U(1) = \mathbf{S}^1$, that is $\mathbf{CP}^n = \mathbf{S}^{2n+1}/U(1) = \mathbf{S}^{2n+1}/\mathbf{S}^1$, since every line in \mathbf{C}^{n+1} intersects the unit sphere in a circle.

II. GRASSMANN VARIETY

In this section, we will define the Grassmann variety. However, the standard reference on geometric invariant theory is [12]. Let $\text{Gr}(r, d)$ be the Grassmann variety of the $r - 1$ -dimensional linear projective subspaces of \mathbf{CP}^{d-1} . Now, we can embed $\text{Gr}(r, d)$ into $\mathbf{P}(\bigwedge^r(\mathbf{C}^d)) = \mathbf{CP}^{\mathcal{N}}$, $\mathcal{N} = \binom{d}{r} - 1$, by using the Plücker map $L \rightarrow \bigwedge^r(L)$, where the exterior product $\bigwedge^r(\mathbf{C}^d)$ for $1 \leq r \leq d$ is a subspace of $\mathbf{C}^{N_1} \otimes \cdots \otimes \mathbf{C}^{N_m}$, spanned by the anti-symmetric tensors. The Plücker coordinates P_{i_1, i_2, \dots, i_r} , $1 \leq i_1 < \cdots < i_r \leq d$ are the projective coordinates in this projective space. Next,

let $\mathbf{C}[\Lambda(r, d)]$ be a polynomial ring with the Plücker coordinates P_J indexed by elements of the set $\Lambda(r, d)$ of ordered r -tuples in $\{1, 2, \dots, d\}$ as its variables. Then the image of the map $\kappa : \mathbf{C}[\Lambda(r, d)] \rightarrow \text{Pol}(\text{Mat}_{r,d})$, which assigns $P_{i_1 i_2 \dots i_r}$ the bracket polynomial $[i_1, i_2, \dots, i_r]$ (the bracket function on the $\text{Mat}_{r,d}$, whose values on a given matrix is equal to the maximal minor formed by the columns from a set of $\{1, 2, \dots, d\}$) is equal to the subring of the invariant of the polynomials. Moreover, the kernel $\mathcal{I}_{r,d}$ of the map κ is equal to the homogeneous ideal of the Grassmann in its Plücker embedding. Furthermore, the homogeneous ideal $\mathcal{I}_{r,d}$ defining $\text{Gr}(r, d)$ in its Plücker embedding is generated by the quadratic polynomials

$$\mathcal{P}_{I,J} = \sum_{t=1}^{r+2} (-)^t P_{i_1, \dots, i_{r-1}, j_t} P_{j_1 \dots j_{t-1} j_{t+1}, \dots, j_{r+1}}, \quad (3)$$

where $I = (i_1 \dots i_{r-1}), 1 \leq i_1 < \dots < i_{r-1} < j_i$, and $J = (j_1, \dots, j_{r+1}), 1 \leq j_1 < \dots < j_{r+1} \leq d$ are two increasing sequences of numbers from the set $\{1, 2, \dots, d\}$. Note that the equations $\mathcal{P}_{I,J} = 0$ define the Grassmannian $\text{Gr}(r, d)$ are called the Plücker coordinate equations. For example, for $\text{Gr}(2, d)$ and $n = 2$, we have

$$\begin{aligned} \mathcal{P}_{I,J} &= \sum_{t=1}^4 (-)^t P_{i_1, j_t} P_{j_1 \dots j_{t-1} j_{t+1}, \dots, j_3} \\ &= -P_{i_1, j_1} P_{j_2, j_3} + P_{i_1, j_2} P_{j_1, j_3} - P_{i_1, j_3} P_{j_1, j_2}, \end{aligned} \quad (4)$$

where $I = (i_1)$, and $J = (j_1, j_2, j_3)$. Note that, by its construction, the Grassmannian $\text{Gr}(2, d)$ is invariant under $SL(2, \mathbf{C})$.

III. PLÜCKER COORDINATES AND MULTIPARTITE ENTANGLEMENT

In this section, we will construct entanglement monotones based on Plücker coordinate equations of the Grassmannian. Let us consider a quantum system $\mathcal{Q}_m^p(2, 2, \dots, 2)$ and let

$$\begin{aligned} \mathcal{E}_{I,J}(\text{Mat}_{r,d}^j) &= \sum_{t=1}^{r+2} (P_j^{i_1, \dots, i_{r-1}, j_t} \overline{P}_{i_1, \dots, i_{r-1}, j_t}^j \\ &\quad + P_j^{j_1 \dots j_{t-1} j_{t+1}, \dots, j_{r+1}} \overline{P}_{j_1 \dots j_{t-1} j_{t+1}, \dots, j_{r+1}}^j), \end{aligned} \quad (5)$$

where $I = (i_1 \dots i_{r-1}), 1 \leq i_1 < \dots < i_{r-1} < j_i$, and $J = (j_1, \dots, j_{r+1}), 1 \leq j_1 < \dots < j_{r+1} \leq d$ are two increasing sequences of numbers from the set $\{1, 2, \dots, d\}$. For example, for $\text{Gr}(2, d)$ and $r = 2$, that is invariant under $SL(2, \mathbf{C})$, we have

$$\begin{aligned} \mathcal{E}_{I,J}(\text{Mat}_{2,d}^j) &= \sum_{t=1}^4 (P_j^{i_1, j_t} \overline{P}_{i_1, j_t}^j + P_j^{j_1 \dots j_{t-1} j_{t+1}, \dots, j_3} \overline{P}_{j_1 \dots j_{t-1} j_{t+1}, \dots, j_3}^j) \\ &= P_j^{i_1, j_1} \overline{P}_{i_1, j_1}^j + P_j^{j_2, j_3} \overline{P}_{j_2, j_3}^j + P_j^{i_1, j_2} \overline{P}_{i_1, j_2}^j \\ &\quad + P_j^{j_1, j_3} \overline{P}_{j_1, j_3}^j + P_j^{i_1, j_3} \overline{P}_{i_1, j_3}^j + P_j^{j_1, j_2} \overline{P}_{j_1, j_2}^j, \end{aligned} \quad (6)$$

where $I = (i_1)$, and $J = (j_1, j_2, j_3)$. Now, we can write the coefficient of a general multi-qubit state as follows

$$\begin{aligned} \text{Mat}_{2,d}^1 &= \begin{pmatrix} \alpha_{1,1,\dots,1} & \alpha_{1,1,\dots,2} & \dots & \alpha_{1,2,\dots,2} \\ \alpha_{2,1,\dots,1} & \alpha_{2,1,\dots,2} & \dots & \alpha_{2,2,\dots,2} \end{pmatrix}, \\ \text{Mat}_{2,d}^2 &= \begin{pmatrix} \alpha_{1,1,\dots,1} & \alpha_{1,1,\dots,2} & \dots & \alpha_{2,1,\dots,2} \\ \alpha_{1,2,\dots,1} & \alpha_{1,2,\dots,2} & \dots & \alpha_{2,2,\dots,2} \end{pmatrix}, \\ &\vdots \\ \text{Mat}_{2,d}^m &= \begin{pmatrix} \alpha_{1,1,\dots,1} & \alpha_{1,1,\dots,1} & \dots & \alpha_{2,2,\dots,1} \\ \alpha_{1,1,\dots,2} & \alpha_{1,1,\dots,2} & \dots & \alpha_{2,2,\dots,2} \end{pmatrix}, \end{aligned} \quad (7)$$

where $d = 2^{m-1}$ and $\text{Mat}_{2,d}^j$, which we get by permutation of j for $1 \leq j \leq m$. Moreover, we assume that the sequences I, J denote the columns of the $\text{Mat}_{2,d}^j$. Then we can define entanglement monotones for the multi-qubit states by

$$\mathcal{E}(\mathcal{Q}_m^p(2, 2, \dots, 2)) = \left(\mathcal{N} \sum_{j=1}^m \mathcal{E}_{I,J}(\text{Mat}_{2,2^{m-1}}^j) \right)^{1/2}. \quad (8)$$

As an example, let us consider the quantum system $\mathcal{Q}_3^p(2, 2, 2)$. For such three-qubit states, if e.g., the subsystem \mathcal{Q}_1 is unentangled with the $\mathcal{Q}_2\mathcal{Q}_3$ subsystems, then the separable set of this state is generated by the six 2-by-2 subdeterminants of

$$\text{Mat}_{2,4}^1 = \begin{pmatrix} \alpha_{1,1,1} & \alpha_{1,1,2} & \alpha_{1,2,1} & \alpha_{1,2,2} \\ \alpha_{2,1,1} & \alpha_{2,1,2} & \alpha_{2,2,1} & \alpha_{2,2,2} \end{pmatrix}. \quad (9)$$

$\text{Mat}_{2,4}^2$ and $\text{Mat}_{2,4}^3$ can be obtained in similar way. Then the partial entanglement monotones for $\text{Mat}_{2,4}^1$ is given by

$$\begin{aligned} \mathcal{E}_{I,J}(\text{Mat}_{2,d}^1) &= P_1^{i_1,j_1} \bar{P}_{i_1,j_1}^1 + P_1^{j_2,j_3} \bar{P}_{j_2,j_3}^1 + P_1^{i_1,j_2} \bar{P}_{i_1,j_2}^1 + P_1^{j_1,j_3} \bar{P}_{j_1,j_3}^1 \\ &\quad + P_1^{i_1,j_3} \bar{P}_{i_1,j_3}^1 + P_1^{j_1,j_2} \bar{P}_{j_1,j_2}^1, \end{aligned} \quad (10)$$

where the Plücker coordinates for $\text{Mat}_{2,4}^1$ are given by

$$\begin{aligned} P_{1,2}^1 &= \alpha_{1,1,1}\alpha_{2,1,2} - \alpha_{1,1,2}\alpha_{2,1,1}, \quad P_{1,3}^1 = \alpha_{1,1,1}\alpha_{2,2,1} - \alpha_{1,2,1}\alpha_{2,1,1}, \\ P_{1,4}^1 &= \alpha_{1,1,1}\alpha_{2,2,2} - \alpha_{1,2,2}\alpha_{2,1,1}, \quad P_{2,3}^1 = \alpha_{1,1,2}\alpha_{2,2,1} - \alpha_{1,2,1}\alpha_{2,1,2}, \\ P_{2,4}^1 &= \alpha_{1,1,2}\alpha_{2,2,2} - \alpha_{1,2,2}\alpha_{2,1,2}, \quad P_{3,4}^1 = \alpha_{1,2,1}\alpha_{2,2,2} - \alpha_{1,2,2}\alpha_{2,2,1}. \end{aligned}$$

Thus, entanglement monotones for three-qubit states is given by

$$\mathcal{E}(\mathcal{Q}_m^p(2, 2, 2)) = \left(\mathcal{N} \sum_{j=1}^3 \mathcal{E}_{I,J}(\text{Mat}_{2,4}^j) \right)^{1/2}. \quad (11)$$

Moreover, for matrices $\text{Mat}_{2,4}^j$, we have $\mathcal{P}_{I,J}^j = -P_{1,2}^j P_{3,4}^j + P_{1,3}^j P_{2,4}^j - P_{1,4}^j P_{2,3}^j = 0$. For three-qubit states, this result coincides with construction of the Segre variety [6]. However, multi-qubit states needs further investigation. Now, as

an example, let us consider the state $|\Psi_W\rangle = \alpha_{1,1,2}|1, 1, 2\rangle + \alpha_{1,2,1}|1, 2, 1\rangle + \alpha_{2,1,1}|2, 1, 1\rangle$. Then we have

$$\mathcal{C}(\mathcal{Q}_3(2, 2, 2)) = (2\mathcal{N}[|\alpha_{1,2,1}\alpha_{2,1,1}|^2 + |\alpha_{1,1,2}\alpha_{2,1,1}|^2 + |\alpha_{1,1,2}\alpha_{1,2,1}|^2])^{1/2}.$$

In particular, for $\alpha_{1,1,2} = \alpha_{1,2,1} = \alpha_{2,1,1} = \frac{1}{\sqrt{3}}$, we get $\mathcal{C}(\mathcal{Q}_3(2, 2, 2)) = (\frac{2}{3}\mathcal{N})^{1/2}$.

IV. HYPERDETERMINANT AND PLÜCKER COORDINATE EQUATIONS

In this section, we will review some result of the construction of entanglement measure based on the hyperdeterminant for three-qubit states and relation between the Plücker coordinate equations and the hyperdeterminant. We also discuss a generalization of this construction. The hyperdeterminant of the elements of $\mathbf{C}^{N_1} \otimes \mathbf{C}^{N_2} \otimes \dots \otimes \mathbf{C}^{N_m}$ was introduced by Gelfand, Kapranov, and Zelevinsky in [13]. They proved that the dual variety of Segre product $\mathbf{CP}^{N_1-1} \times \mathbf{CP}^{N_2-1} \times \dots \times \mathbf{CP}^{N_m-1}$ is a hypersurface if and only if $N_j \leq \sum_{i \neq j} N_i$ for $j = 1, 2, \dots, m$. Whenever the dual variety is a hypersurface its equation is called the hyperdeterminant of the format $N_1 \times N_2 \times \dots \times N_m$ and denoted by Det . The hyperdeterminant is a homogeneous polynomial function over $\mathbf{C}^{N_1} \otimes \mathbf{C}^{N_2} \otimes \dots \otimes \mathbf{C}^{N_m}$, so that the condition $\text{Det}A \neq 0$ is meaningful for $A \in \mathbf{CP}^{N_1 N_2 \dots N_m - 1}$. Moreover, the hyperdeterminant Det is $SL(N_1, \mathbf{C}) \times SL(N_2, \mathbf{C}) \otimes \dots \otimes SL(N_m, \mathbf{C})$ -invariant. For example, for $m = 2$ we have $\text{Det}A = \alpha_{1,1}\alpha_{2,2} - \alpha_{1,2}\alpha_{2,1}$ and for $m = 3$, we have

$$\begin{aligned} \text{Det}A = & \alpha_{1,1,1}^2 \alpha_{2,2,2}^2 + \alpha_{1,1,2}^2 \alpha_{2,2,1}^2 + \alpha_{1,2,1}^2 \alpha_{2,1,2}^2 + \alpha_{2,1,1}^2 \alpha_{1,2,2}^2 \\ & - 2(\alpha_{1,1,1}\alpha_{1,1,2}\alpha_{2,2,1}\alpha_{2,2,2} + \alpha_{1,1,1}\alpha_{1,2,1}\alpha_{2,1,2}\alpha_{2,2,2} \\ & + \alpha_{1,1,1}\alpha_{2,1,1}\alpha_{1,2,2}\alpha_{2,2,2} + \alpha_{1,1,2}\alpha_{1,2,1}\alpha_{2,1,2}\alpha_{2,2,1} \\ & + \alpha_{1,1,2}\alpha_{2,1,1}\alpha_{1,2,2}\alpha_{2,2,1} + \alpha_{1,2,1}\alpha_{2,1,1}\alpha_{1,2,2}\alpha_{2,1,2}) \\ & + 4(\alpha_{1,1,1}\alpha_{2,2,1}\alpha_{2,1,2}\alpha_{1,2,2} + \alpha_{2,2,2}\alpha_{2,1,1}\alpha_{1,2,1}\alpha_{1,1,2}). \end{aligned} \quad (12)$$

Now, let us introduce the Diophantine function. For any sequence of number $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, and $\delta_1, \delta_2, \delta_3, \delta_4$ we have

$$\mathcal{P}(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \delta_1, \delta_2, \delta_3, \delta_4) = (\gamma_1\delta_1 + \gamma_2\delta_2 - \gamma_3\delta_3 - \gamma_4\delta_4)^2 - 4(\gamma_1\gamma_2 + \delta_3\delta_4)(\gamma_3\gamma_4 + \delta_1\delta_2). \quad (13)$$

This equation is equal to the hyperdeterminant $\text{Det}A$. For the quantum system $\mathcal{Q}_3^p(2, 2, 2)$ we have

$$\text{Det}A = \mathcal{P}(-\alpha_{1,1,1}, \alpha_{2,2,1}, \alpha_{2,1,2}, \alpha_{1,2,2}, -\alpha_{2,2,2}, \alpha_{1,1,2}, \alpha_{1,2,1}, \alpha_{2,1,1}). \quad (14)$$

For the quantum system $\mathcal{Q}_3^p(2, 2, 2)$, one can wonder if it would be possible to find a generalization of the polynomial \mathcal{P} such that the hyperdeterminant $\text{Det}A$ could be give by \mathcal{P} . We can also construct the hyperdeterminant in terms of the Plücker coordinates. For example, P. Lévy [5] has constructed such Plücker coordinates for three-qubit states as follows. Let $\alpha^p, \beta^q, p, q = 1, 2, 3, 4$ be two-four component vectors defined as $\alpha_{1,k_1,k_2} = \frac{1}{\sqrt{2}}\alpha_p \Sigma^{p,k_1,k_2}$, and $\alpha_{2,k_1,k_2} = \frac{1}{\sqrt{2}}\beta_p \Sigma^{p,k_1,k_2}$, where $\Sigma^s = -i\sigma_s, s = 1, 2, 3$, and $\Sigma^4 = I_2$, where σ_p are Pauli matrices. Then the hyperdeterminant is

given by $\text{Det}\mathcal{A} = 2P_{p,q}P^{p,q}$, where the Plücker coordinates are given by $P_{p,q} = \alpha_p\beta_q - \alpha_q\beta_p$. Thus, this construction can be at least extended into multi-qubit states following our definition of the Plücker coordinates for the multi-qubit states and further extending the definition of two 2^{m-1} components vectors $\alpha^p, \beta^q, p, q = 1, 2, \dots, 2^{m-1}$. For further progress in this direction see Ref. [8].

V. CONCLUSION

In this paper, we have constructed entanglement monotones for multipartite states based on the Grassmannian $\text{Gr}(r, d)$, which was defined in terms of the Plücker coordinate equations. In particular, we have given an explicit expression for entanglement monotones for multi-qubit states. Moreover, we have investigated entanglement monotones for three-qubit state as an illustrative example.

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